RADIATIVE BACKREACTION
ON GLOBAL STRINGS

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Abstract

We consider radiative backreaction for global strings using the Kalb-Ramond formalism. In analogy to the point electron in classical electrodynamics, we show how local radiative corrections to the equations of motion allow one to remove the divergence in the self field and calculate a first order approximation to the radiation backreaction force. The effects of this backreaction force are studied numerically by resubstituting the equations of motion to suppress exponentially growing solutions. By direct comparison with numerical field theory simulations and analytic radiation calculations we establish that the 'local backreaction approximation' provides a satisfactory quantitative description of radiative damping for a wide variety of string configurations. Finally, we discuss the relevance of this work to the evolution of a network of global strings and their possible cosmological consequences. These methods can also be applied to describe the effects of gravitational radiation backreaction on local strings, electromagnetic radiation backreaction on superconducting strings and other forms of string radiative backreaction.

1. Introduction

A variety of unified field theories predict the formation of a network of topological defects at one or more phase transitions in the early universe [1]. Strings associated with

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the breaking of local symmetries have generated the most interest in the literature because, amongst other reasons, GUT-scale strings could have been the initial seeds for the formation of large-scale structure [2]. However, local strings are tightly constrained by their contribution to the gravitational radiation background [3]. There are other types of strings which circumvent this constraint and which may have similar cosmological implications, in particular those formed when a global symmetry is broken. Instead of radiating gravitationally, the dominant radiation mechanism for these strings is the emission of massless Nambu-Goldstone bosons [4]. In a recent publication [5], we studied the nature of this radiation in detail, using analytic and numerical techniques. We demonstrated that a low energy effective action known as the Kalb-Ramond action, provided an accurate description of the dynamics of global strings even at the moderately high velocities one expects in a realistic string network. Within this formalism the topological coupling of the massless field to the string is linearized. One finds that the coupling between the field and the string worldsheet is similar to that of the point electron in electromagnetism. However, there are still difficulties associated with this approach, notably because equations of motion are inconsistent due to a divergent self-field.

This type of problem has been well understood for some time in the context of a point electron in classical electrodynamics [6]. In the case of the electron, the self and radiation fields can be distinguished easily since, at large distances R, the self field falls off as $1/R^2$, whereas the radiation field falls off as $1/R$. Careful analysis of the equations of motion leads to the renormalisation of the electron mass by the Coulomb self-field, using the classical electron radius to cut off short distance divergences and a first order approximation to the radiation backreaction force known as the Abraham-Lorentz force,

$$F_{\mu}^{\rm rad} = \frac{2}{3}\,\frac{e^2}{4\pi}\left(\dddot{X}_{\mu} + \ddot{X}^2\dot{X}_{\mu}\right),\tag{1}$$

where $X_{\mu}(\tau)$ is the position on the electron's worldline at time τ. The dependence of this force on $\dddot{X}_{\mu}$ has lead to problems in numerical applications since there exist exponentially increasing solutions to the equations of motion. These unphysical 'runaway' solutions can only be suppressed by rewriting the equations of motion as an integro-differential

equation.

In a recent letter [7], we proposed a formalism for removing the self force and calculating a first approximation for the radiation backreaction force of strings using the analogy of classical electrodynamics. However, the analogy is not exact because the strings are line-like objects of possibly infinite extent. We circumvent this problem by assuming that the dominant contribution to the backreaction force comes from string segments in the vicinity of the point in question, henceforth known as the 'local backreaction approximation'. This approximation will not be valid in every situation, but in a wide variety of circumstances it should work well. In this paper we will consider this approximation in greater detail. We will elaborate on the derivation of the radiation backreaction force and give a discussion of the physical aspects of the approximation. Further numerical evidence will be presented in support of the validity of this approximation in physically important cases by direct comparison between modified Nambu dynamics, evolved numerically using the backreaction force, and numerical field theory simulations. The one free parameter in our analysis, effectively the damping coefficient, can be normalized by comparing with numerical field theory simulations and known analytic results.

One of the main motivations for this work is to implement appropriate radiative corrections in a full network simulation. We anticipate that the scaling assumption for gauge strings, numerically verified in refs. [8,9], will also be seen to be valid for global strings. However, it is anticipated that the parameters quantifying the small scale features will be somewhat different [10]. Accurate numerical simulations will allow estimates of the cosmological axion density to be refined. A similar formalism is applicable to gravitational radiation backreaction on local strings [11] and electromagnetic radiation backreaction on superconducting strings.

Throughout this paper we employ a $(+ - - -)$ signature for the spacetime metric $g_{\mu\nu}$ and $(+ -)$ for the induced metric on the string worldsheet γ_{ab}, the coordinates for which are given by $X^\mu = X^\mu(\sigma, \tau)$, with the null coordinates, $u = \sigma - \tau$, $v = \sigma + \tau$.

2. Analytic formalism

2.1 The Kalb-Ramond action

Th essential features of global strings in flat space are exhibited in the simple U(1) Goldstone model, with action given by

$$S = \int d^4x \left[\partial_\mu \bar{\Phi} \partial^\mu \Phi - \frac{1}{4} \lambda (\bar{\Phi}\Phi - f_{\rm a}^2)^2 \right], \tag{2}$$

where $\Phi = \phi e^{i\vartheta}$ is a complex scalar field which can be split into a massive (real) component ϕ and a massless (real periodic) Goldstone boson ϑ. The analytic treatment of global string dynamics is hampered by the topological coupling of the self field of the string to the Goldstone boson radiation field. However, we can exploit the well-known duality between a massless scalar field and a two-index antisymmetric tensor $B_{\mu\nu}$ to replace the Goldstone boson ϑ in (2) via the relation

$$\phi^2 \partial_\mu \vartheta = \frac{1}{2} f_{\rm a} \epsilon_{\mu\nu\lambda\rho} \partial^\nu B^{\lambda\rho}. \tag{3}$$

Performing this transformation carefully and integrating over the massive degrees of freedom about the two-dimensional string worldsheet $X^\mu(\sigma, \tau)$ [12], yields the flat-space Kalb–Ramond action [13,14],

$$S = -\mu_0 \int d\sigma d\tau \sqrt{-\gamma} + \frac{1}{6} \int d^4x H^2 - 2\pi f_{\rm a} \int B_{\mu\nu} d\sigma^{\mu\nu}, \tag{4}$$

where $H_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta} + \partial_\beta B_{\mu\alpha} + \partial_\alpha B_{\beta\mu}$ is the field strength of the antisymmetric tensor field $B_{\mu\nu}$, the metric induced on the world sheet is

$$\gamma_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \quad \gamma = \det(\gamma_{ab}), \tag{5}$$

and the area element on the worldsheet is

$$d\sigma^{\mu\nu} = \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu d\sigma d\tau.$$

The first term is the familar Nambu action for local strings, the second is the antisymmetric field strength for both external fields and the self-field of the string and the

last term is a contact interaction between the antisymmetric tensor field and the string worldsheet. The coupling between the string and the antisymmetric tensor is analogous to the electromagnetic coupling of the point electron to the electromagnetic field. This analogy underpins our subsequent development of global string dynamics based on (4).

Varying the action (4) with respect to the worldsheet coordinates and the antisymmetric tensor yields the string equations of motion and the tensor field equations,

$$\mu_0 \partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) = \mathcal{F}^\mu = 2\pi f_{\mathrm{a}} H^{\mu\alpha\beta} V_{\alpha\beta}\,,$$

$$\partial_\mu H^{\mu\alpha\beta} = -4\pi J^{\alpha\beta} = -2\pi f_{\mathrm{a}} \int d\sigma d\tau \delta^4\bigl(x - X(\sigma,\tau)\bigr) V^{\alpha\beta}\,, \tag{6}$$

where $V_{\alpha\beta} = \epsilon^{ab}\partial_a X_\alpha \partial_b X_\beta$ is the antisymmetric vertex operator. In the conformal string gauge and the Lorentz antisymmetric tensor gauge,

$$\dot{X}^2 + X'^2 = 0\,, \quad \dot{X}\cdot X' = 0\,, \quad \partial_\mu B^{\mu\nu} = 0\,, \tag{7}$$

the equations of motion (6) become

$$\mu_0\bigl(\ddot{X}^\mu - X''^\mu\bigr) = \mathcal{F}^\mu = 2\pi f_{\mathrm{a}} H^{\mu\alpha\beta} V_{\alpha\beta}\,,$$

$$\Box B_{\alpha\beta} = -4\pi J_{\alpha\beta} = -2\pi f_{\mathrm{a}} \int d\sigma d\tau \delta^4\bigl(x - X(\sigma,\tau)\bigr) V_{\alpha\beta}\,, \tag{8}$$

where $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$ and $V_{\alpha\beta} = \dot{X}_\alpha X'_\beta - X'_\alpha \dot{X}_\beta$. These equations are problematic because the self-field diverges as any point of the string is approached, that is $x \to X(\sigma,\tau)$.

2.2 Simple string configurations

If one ignores the effects of the force density $\mathcal{F}^\mu$, then the equations of motion reduce to the well-known Nambu equations of motion, a massless wave equation. The equations have solution

$$X^0 = t = \tau\,, \quad \mathbf{X} = \frac{1}{2}\bigl[\mathbf{a}(u) + \mathbf{b}(v)\bigr]\,, \tag{9}$$

where the functions $\mathbf{a}(u)$ and $\mathbf{b}(v)$ are the left- and right-moving parts of the solution (recall $u = \sigma - t$ and $v = \sigma + t$). Using the conformal gauge conditions (7), one can deduce that

$$\mathbf{a}'^2 = 1\,, \quad \mathbf{b}'^2 = 1\,. \tag{10}$$

Figure 1: Schematic of the solutions of the Nambu equations of motion: (a) loop solutions parametrized by the invariant length L and (b) long string solutions parameterized by the wavelength L and the relative amplitude $\mathcal{E}$.

The equations (9) and (10) have closed loop and periodic long (or infinite) string solutions. The loop solutions are parametrized by the length of the loop L, which is closely related to the characteristic frequency $\Omega = 2\pi/L$, whereas the long periodic solutions are parametrized by the wavelength L and the ratio of amplitude to wavelength or the relative amplitude $\mathcal{E} = 2\pi A/L$, where A is the amplitude. Fig. 1 shows a schematic of the two types of solution we shall consider. In general situations, such solutions will correspond to a superposition of a large number of harmonics.

A simple two parameter family of loops, known as Kibble-Turok loops, involve just the first and third harmonics [15]:

$$
\begin{aligned}
\mathbf{X} = \frac{1}{2\Omega}\Big(& (1-\alpha)\sin\Omega u + \tfrac{1}{3}\alpha\sin 3\Omega u + \sin\Omega v, \\
& - (1-\alpha)\cos\Omega u - \tfrac{1}{3}\alpha\cos 3\Omega u - \cos\psi\cos\Omega v, \\
& - 2\big[\alpha(1-\alpha)\big]^{1/2}\cos\Omega u - \sin\psi\cos\Omega v\Big),
\end{aligned}
\tag{11}
$$

where $\Omega = 2\pi/L$ and $0 \leq \alpha \leq 1, -\pi \leq \psi \leq \pi$. If $\alpha = 0$ and $\psi = 0$ then the solution is a circular loop, which oscillates between a circle of radius $L/2\pi$ and a point. For a significant range of the parameters α and ψ these solutions can be shown to be non

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self-intersecting and so the dominant decay mechanism is likely to be through radiation rather than loop fragmentation. The time evolution of a particular solution with $\psi = \pi/3$ and $\alpha = 0.5$ is shown in fig. 2a. Kibble–Turok loops also generate cusps, that is, points on the string which reach the velocity of light. For example, cusps will appear on the $\alpha = 0$ solution at $\sigma = L/4, 3L/4$ when $t = (n + 1/2)L/2$ (n integer). Whether or not cusps are generic on realistic loops has been the subject of various heuristic discussion, which have also considered the unknown effect of backreaction on cusp evolution.

Strings reconnect or 'exchange partners' when they intersect. This process introduces kinks—contact discontinuities in the velocity $\dot{\mathbf{X}}$ and tangent vector $\mathbf{X}'$—which propagate along the string at the speed of light. Because realistic loops are produced by long string reconnections or self-intersections we can be sure they will possess at least two kinks, probably more. An idealized loop with four kinks between four straight string segments is given by the following [16]

$$\mathbf{X} = \frac{1}{2}\big[\mathbf{a}(\sigma - \tau) + \mathbf{b}(\sigma + \tau)\big],$$

where

$$\mathbf{a}(\sigma - \tau) = \begin{cases} \left(\dfrac{L}{2\pi}(\sigma - \tau) - \dfrac{L}{4}\right)\mathbf{A} & 0 < \sigma - \tau < \pi \\[2mm] \left(\dfrac{3L}{4} - \dfrac{L}{2\pi}(\sigma - \tau)\right)\mathbf{A} & \pi < \sigma - \tau < 2\pi, \end{cases} \tag{12}$$

$$\mathbf{b}(\sigma + \tau) = \begin{cases} \left(\dfrac{L}{2\pi}(\sigma + \tau) - \dfrac{L}{4}\right)\mathbf{B} & 0 < \sigma + \tau < \pi \\[2mm] \left(\dfrac{3L}{4} - \dfrac{L}{2\pi}(\sigma + \tau)\right)\mathbf{B} & \pi < \sigma + \tau < 2\pi, \end{cases} \tag{13}$$

with arbitrary unit vectors $\mathbf{A}$ and $\mathbf{B}$. The two pairs of kinks propagate in opposite directions around the loop. In the special case $\mathbf{A} \cdot \mathbf{B}$, the loop is planar and oscillates between a square and a doubled line.

A simple, symmetric long string solution can be constructed from equal and oppositely propagating helicoidal waves in the fundamental mode [17],

$$\begin{aligned} \mathbf{X} &= \left(\frac{\mathcal{E}}{2\Omega}\big[\cos\Omega u + \cos\Omega v\big], \frac{\mathcal{E}}{2\Omega}\big[\sin\Omega u + \sin\Omega v\big], \frac{1}{2}\sqrt{1 - \mathcal{E}^2}\,(u + v)\right) \\ &= \left(\frac{\mathcal{E}}{\Omega}\sin\Omega\sigma\cos\Omega t, \frac{\mathcal{E}}{\Omega}\cos\Omega\sigma\cos\Omega t, \sqrt{1 - \mathcal{E}^2}\sigma\right), \end{aligned} \tag{14}$$

where $0 < \mathcal{E} < 1$ and $\mathcal{E} \to 1$ in the relativistic limit. This corresponds to a helicoidal solution which oscillates between a static helix and a straight line, as shown in fig. 2(b) for $\mathcal{E} = 0.6$. Because of its perfect symmetry, calculations for the radiation power from this solution are analytically tractable [17,5], though such configurations are unlikely to be found in a realistic string network.

One can generalise the helicoidal solution (14) to have unequal left and right moving amplitudes,

$$\mathbf{X} = \left(\frac{\mathcal{E}_R}{2\Omega}\cos\Omega u + \frac{\mathcal{E}_L}{2\Omega}\cos\Omega u, \frac{\mathcal{E}_R}{2\Omega}\sin\Omega u + \frac{\mathcal{E}_L}{2\Omega}\sin\Omega u, \frac{1}{2}\sqrt{1 - \mathcal{E}_R^2}\,u + \frac{1}{2}\sqrt{1 - \mathcal{E}_L^2}\,v \right), \quad (15)$$

where $0 < \mathcal{E}_R, \mathcal{E}_L < 1$. This type of solution is thought to be a reasonably accurate description of long strings in a realistic network, since within a sufficiently small volume the number of left and right movers are unlikely to be strongly correlated. One special case of this solution is that with no left moving perturbation,

$$\mathbf{X} = \left(\frac{\mathcal{E}_R}{2\Omega}\cos\Omega u, \frac{\mathcal{E}_R}{2\Omega}\sin\Omega u, \frac{1}{2}\sqrt{1 - \mathcal{E}_R^2}\,u + \frac{1}{2}v \right). \quad (16)$$

It has been suggested that pure left- or right-moving configurations do not radiate and so they will propagate indefinitely [18]. However, we shall argue that such solutions are not physically relevant because they require initial data with the string fields artificially correlated out to infinity.

A solution, similar to the helix (14), but with sinusoidal perturbations in only one plane is

$$\mathbf{X} = \left(\frac{\mathcal{E}}{2\Omega}\left[\cos\Omega u + \cos\Omega v\right], 0, \frac{1}{2\Omega}\left[E(\mathcal{E},\Omega u) + E(\mathcal{E},\Omega v)\right] \right), \quad (17)$$

where $E(k,\phi)$ is the incomplete elliptic integral of the second kind, defined by

$$E(k,\phi) = \int_0^\phi d\theta\sqrt{1 - k^2\sin^2\theta}, \quad (18)$$

where $0 < \mathcal{E} < 1$ and $\mathcal{E} \to 1$ in the relativistic limit.

Long string solutions of (10) can also possess kinks as in (12). A simple solution for a periodic distribution of kinks on a perturbed straight string, consists of the left- and

right-moving perturbations, X_U and X_V respectively,

$$\mathbf{X} = \left(X_U + X_V, 0, \frac{1}{2}\sqrt{1 - \frac{4\mathcal{E}^2}{\pi^2}}(u + v)\right),\tag{19}$$

where

$$X_U = \begin{cases} \dfrac{2\mathcal{E}}{\pi}u & 0 < u < \dfrac{1}{4}L \\[2mm] \dfrac{2\mathcal{E}}{\pi}\left(\dfrac{1}{2}L - u\right) & \dfrac{1}{4}L < u < \dfrac{3}{4}L \\[2mm] \dfrac{2\mathcal{E}}{\pi}\left(-L + u\right) & \dfrac{3}{4}L < u < L\,, \end{cases}\tag{20}$$

$$X_V = \begin{cases} \dfrac{2\mathcal{E}}{\pi}v & 0 < v < \dfrac{1}{4}L \\[2mm] \dfrac{2\mathcal{E}}{\pi}\left(\dfrac{1}{2}L - v\right) & \dfrac{1}{4}L < v < \dfrac{3}{4}L \\[2mm] \dfrac{2\mathcal{E}}{\pi}\left(-L + v\right) & \dfrac{3}{4}L < v < L\,, \end{cases}\tag{21}$$

where $0 < \mathcal{E} < \frac{\pi}{2}$ and, in this case, $\mathcal{E} \to \frac{\pi}{2}$ is the relativistic limit. In fig. 2d, notice how the kinks split into two which propagate at the speed of light in opposite directions along the string.

2.3 Green functions

One of the most basic techniques of mathematical physics is the inversion of differential equations such as the field equation in (8) using Green functions. The basic Green functions satisfy

$$\Box D(x) = \delta^4(x)\,,\tag{22}$$

which implies that the solution to

$$\Box F(x) = S(x)\,,\tag{23}$$

is given by integrating the product of the Green function and the forcing term $S(x)$ over spacetime

$$F(x) = \int d^4x' D(x - x')S(x')\,.\tag{24}$$

In order to deduce a specific form for the Green function one must specify some boundary conditions which define the region over which initial data is known. The two

Figure 2: The evolution of various string trajectories: (a) Kibble-Turok loop (11) ($\psi = \pi/3$, $\alpha = 0.5$), (b) four kink loop solution (12) (c) helicoidal long string (14) with ($\mathcal{E} = 0.6$), and (d) kink solution (19) with $45°$ openning angle.

most common Green functions used are the retarded and advanced time Green functions, which use initial data on the backward or forward light cones respectively,

$$D_{\text{ret}}(x) = \frac{1}{2\pi}\theta(x^0)\delta(x^2)\,, \quad D_{\text{adv}}(x) = \frac{1}{2\pi}\theta(-x^0)\delta(x^2)\,, \tag{25}$$

where $x^\mu = (x^0, \mathbf{x})$ and $\theta(x^0)$ is the Heaviside function, that is $\theta(x^0) = 1$ for $x^0 > 0$ and $\theta(x^0) = 0$ otherwise.

In problems where radiation is involved one wishes to separate radiative effects from those of the self-field. The radiation field is free and the radiation Green function must satisfy a homogeneous version of (22). One can construct such a Green function by

subtracting the advanced Green function from the retarded. Similarly, one can construct the Green function for the self-field by summing the retarded and advanced Green's functions. Using appropriate normalisations, one can deduce that

$$
\begin{aligned}
D_{\mathrm{rad}} &= \frac{1}{2}\left(D_{\mathrm{ret}} - D_{\mathrm{adv}}\right) = \frac{1}{4\pi}\epsilon(x^0)\delta(x^2)\,, \\
D_{\mathrm{self}} &= \frac{1}{2}\left(D_{\mathrm{ret}} + D_{\mathrm{adv}}\right) = \frac{1}{4\pi}\delta(x^2)\,,
\end{aligned}
\tag{26}
$$

where $\epsilon(x^0) = \theta(x^0) - \theta(-x^0)$. One can calculate the self- and radiation-field for a problem such as (23) using similar expression to (24) with the basic Green function replaced by the appropriate expression from (26).

2.4 Lienard-Wiechart Potentials

Using the Green function techniques described in the previous section, one can deduce that

$$
\begin{aligned}
B_{\alpha\beta}(x) &= -4\pi \int d^4x'\, D_{\mathrm{ret}}(x - x')J_{\alpha\beta}(x') \\
&= -2\pi f_{\mathrm{a}} \int d\bar{\sigma}d\bar{\tau}\, D_{\mathrm{ret}}(x - X(\bar{\sigma},\bar{\tau}))V_{\alpha\beta}(\bar{\sigma},\bar{\tau})\,.
\end{aligned}
\tag{27}
$$

The integration is over all time and over all string segments. In the case of a closed loop this is a finite range, but for long (infinite) strings the range is infinite. If one defines $\Delta_\mu = x_\mu - X_\mu(\bar{\sigma},\bar{\tau})$ while treating $\bar{\sigma}$ as $\bar{\sigma} = \bar{\sigma}(\bar{\tau})$, then

$$
d(\Delta^2) = -2\Delta \cdot \dot{X}d\bar{\tau}\,,
$$

$$
\partial_\rho = 2\Delta_\rho \frac{\partial}{\partial(\Delta^2)} = -\frac{\Delta_\rho}{\Delta \cdot \dot{X}}\frac{\partial}{\partial\bar{\tau}}\,.
$$

Substituting into (27) and evaluating the delta function one can deduce the Lienard-Wiechart potential [6,19,20]

$$
B_{\alpha\beta}(x) = -\frac{f_{\mathrm{a}}}{2} \int d\bar{\sigma}\left(\frac{V_{\alpha\beta}}{|\Delta.\dot{X}|}\right)\Bigg|_{\bar{\tau}=\tau_R}\,,
\tag{28}
$$

where $\Delta^2|_{\bar{\tau}=\tau_R} = 0$ and $\tau_R < t$. The modulus sign in (28) preserves the orientation of the region of integration when evaluating the delta-function. In order to calculate the radiation backreaction force one requires the derivative of (28). This can be calculated by performing an integration by parts,

$$
\partial_\mu B_{\alpha\beta}(x) = -\frac{f_{\mathrm{a}}}{2} \int d\bar{\sigma}\frac{1}{\Delta.\dot{X}}\frac{\partial}{\partial\bar{\tau}}\left(\frac{\Delta_\mu V_{\alpha\beta}}{|\Delta.\dot{X}|}\right)\Bigg|_{\bar{\tau}=\tau_R}\,.
\tag{29}
$$

One can separate the radiation field from the self field by using the Green functions for the self- and radiation-fields (26). Therefore, one can calculate the Lienard-Wiechart potentials and their derivatives for both the self and radiation fields.

$$B_{\alpha\beta}^{\text{self}}(x) = -\frac{f_{\text{a}}}{4}\int d\bar\sigma \left[\left.\frac{V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right|_{\bar\tau=\tau_R} + \left.\frac{V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right|_{\bar\tau=\tau'_R}\right],$$

$$B_{\alpha\beta}^{\text{rad}}(x) = -\frac{f_{\text{a}}}{4}\int d\bar\sigma \left[\left.\frac{V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right|_{\bar\tau=\tau_R} - \left.\frac{V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right|_{\bar\tau=\tau'_R}\right],$$

$$\partial_\mu B_{\alpha\beta}^{\text{self}}(x) = -\frac{f_{\text{a}}}{4}\int d\bar\sigma \left[\left.\frac{1}{\Delta\cdot\dot X}\frac{\partial}{\partial\bar\tau}\left(\frac{\Delta_\mu V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right)\right|_{\bar\tau=\tau_R} + \left.\frac{1}{\Delta\cdot\dot X}\frac{\partial}{\partial\bar\tau}\left(\frac{\Delta_\mu V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right)\right|_{\bar\tau=\tau'_R}\right],$$

$$\partial_\mu B_{\alpha\beta}^{\text{rad}}(x) = -\frac{f_{\text{a}}}{4}\int d\bar\sigma \left[\left.\frac{1}{\Delta\cdot\dot X}\frac{\partial}{\partial\bar\tau}\left(\frac{\Delta_\mu V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right)\right|_{\bar\tau=\tau_R} - \left.\frac{1}{\Delta\cdot\dot X}\frac{\partial}{\partial\bar\tau}\left(\frac{\Delta_\mu V_{\alpha\beta}}{|\Delta\cdot\dot X|}\right)\right|_{\bar\tau=\tau'_R}\right].$$
$$(30)$$

where $\Delta^2|_{\bar\tau=\tau_R,\tau'_R} = 0$ and $\tau_R < t, \tau'_R > t$. Effectively, then, we have performed the separation of the self- and radiation-fields. An attempt was made to perform this split in ref. [21], using techniques similar to those used in classical electrodynamics [6]. This method performed the split on the basis of asymptotic fall off. As already discussed, this procedure works in the case of the electron since the self-field falls off like $1/R^2$, whereas the radiation field falls off like $1/R$ for large R. However, this procedure may work for string loops. However, it is doomed to failure for long strings, since both the self- and radiation-fields fall off as $1/R$.

2.5 The 'local backreaction approximation'

It has already been noted that the renormalisation procedure for strings is more complicated than that for the point electron. The main problem becomes obvious when one compares the Lienard-Wiechart potentials for strings to those for the electron [6]. Since the string is an extended object, the Lienard-Wiechart potential is an integral along the string. In the case of a loop of length L, this integral will be in the range $0 < |\sigma - \bar\sigma| < L$ for a point $X(\sigma,\tau)$ on the string and can be easily approximated. However, in the case of a long (infinite) string the range is $-\infty < |\sigma - \bar\sigma| < \infty$ and the integral cannot be evaluated without the solution being periodic.

In more general situations this is not possible and one must make what we shall

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call the 'local backreaction approximation'. Since the effects of backreaction from string segments at large distances from the point in question must be suppressed, it seems sensible to truncate the integrals of (28) and (30) at some renormalisation scale Δ, which is at present arbitrary. That is the integrals are over the range $-\Delta/2 < |\sigma - \bar{\sigma}| < \Delta/2$. Using the case of the loop of length L as an example seems to suggest that $\Delta \sim L$. In fact our expectation is that, for more general string trajectories, an appropriate choice for Δ would be near the average curvature radius of the string.

Using the local backreaction approximation one can perform the renormalisation of the self-field and the derivation of a first order approximation to the radiation-field. If one allows $x \to X(\sigma, \tau)$, then (30) can be expanded in terms of $s = \sigma - \bar{\sigma}$ and $t = \tau - \bar{\tau}$. This procedure requires that the natural scale for the otherwise arbitrary renormalisation cut-off Δ be less than the average curvature radius of the string. In this case one finds that the condition $\Delta^2|_{\tau_R} = 0$, implies that $t = |s| + O(s^4)$ and $\Delta^2|_{\tau'_R} = 0$, implies that $t = -|s| + O(s^4)$. Ignoring terms of order four in s and t, allows one to deduce that

$$
\begin{aligned}
H_{\mu\alpha\beta}^{\text{self}} &= \frac{f_{\text{a}}}{2\dot{X}^4}\left[\ddot{X}_{[\mu}V_{\alpha\beta]} - X_{[\mu}''V_{\alpha\beta]}\right]\log(\Delta/\delta) + \mathcal{O}(\Delta^2)\,, \\
H_{\mu\alpha\beta}^{\text{rad}} &= \frac{f_{\text{a}}}{2\dot{X}^4}\left[-\frac{4}{3}\dddot{X}_{[\mu}V_{\alpha\beta]} - \frac{1}{2}\dot{X}_{[\mu}\dot{V}_{\alpha\beta]} + \left(\frac{2\dot{X}.\ddot{X}}{\dot{X}^2}\right)\ddot{X}_{[\mu}V_{\alpha\beta]}\right]\Delta + \mathcal{O}(\Delta^2)\,,
\end{aligned}
\tag{31}
$$

where $A_{[\mu\alpha\beta]} = A_{\mu\alpha\beta} + A_{\beta\mu\alpha} + A_{\alpha\beta\mu}$. Note that the self-field has no order Δ term. Ignoring terms of order Δ^2, we can then obtain expressions for the self-force and the first order approximation to the radiation backreaction force density,

$$
\begin{aligned}
\mathcal{F}_{\mu}^{\text{self}} &= -2\pi f_{\text{a}}^2 \log(\Delta/\delta)\left[\ddot{X}_\mu - X_\mu''\right] \\
\mathcal{F}_{\mu}^{\text{rad}} &= \pi f_{\text{a}}^2 \Delta\left\{\frac{4}{3}\dddot{X}_\mu - 2\left(\frac{\dot{X}.\ddot{X}}{\dot{X}^2}\right)\ddot{X}_\mu + 2\left(\frac{X'.\ddot{X}}{\dot{X}^2}\right)\dot{X}_\mu' + \left[-\frac{4}{3}\left(\frac{\dot{X}.\dddot{X}}{\dot{X}^2}\right) + 2\left(\frac{\dot{X}.\ddot{X}}{\dot{X}^2}\right)^2\right.\right. \\
&\left.\left. + 2\left(\frac{X'.\ddot{X}}{\dot{X}^2}\right)^2\right]\dot{X}_\mu + \left[\frac{4}{3}\left(\frac{X'.\dddot{X}}{\dot{X}^2}\right) - 4\left(\frac{(\dot{X}.\ddot{X})(X'.\ddot{X})}{\dot{X}^2}\right)\right]X_\mu'\right\}\,,
\end{aligned}
\tag{32}
$$

where $\delta\,(<< \Delta)$ is the width of the string core and corresponds to the ultra-violet renormalisation scale. These expressions for the self and radiation force densities are extremely complicated, however, our confidence that these are the correct expression is

strengthened since they non-trivially respect the conformal gauge conditions, that is, $\mathcal{F}^{\text{self}} \cdot \dot{X} = 0$, $\mathcal{F}^{\text{self}} \cdot X' = 0$, $\mathcal{F}^{\text{rad}} \cdot \dot{X} = 0$ and $\mathcal{F}^{\text{rad}} \cdot X' = 0$.

The self-field is a multiple of the left hand side of the equations of motion and facilitates the well-known renormalisation of the string tension, in a way exactly analogous to the mass of a point electron. The equations of motion in this case are

$$\mu(\Delta)\left[\ddot{X}_\mu - X''_\mu\right] = \mathcal{F}^{\text{rad}}_\mu \,, \tag{33}$$

where $\mu(\Delta) = \mu_0 + 2\pi f_{\text{a}}^2 \log(\Delta/\delta)$ is the renormalised string tension

For general string trajectories, similar to those discussed in ref. [5], some of the terms in (32) can be shown to be sub-dominant. In particular, if the string solution is specified by the relative amplitude $\mathcal{E}$ and its wavelength L, then one finds that $\mathbf{X} \sim \mathcal{O}(\mathcal{E}L)$ and each subsequent derivative requires a division by L. Most of the higher order terms in $\mathcal{E}$ can be dropped, though it is necessary to keep two of the higher order terms to maintain the gauge conditions. One then finds that it is possible to approximate $H^{\text{rad}}_{\mu\alpha\beta}$ and $\mathcal{F}^{\text{rad}}_\mu$ by

$$\begin{aligned}
H^{\text{rad}}_{\mu\alpha\beta} &\approx -\frac{2f_{\text{a}}\Delta}{3\dot{X}^4}\dddot{X}_{[\mu}V_{\alpha\beta]} \,, \\
\mathcal{F}^{\text{rad}}_\mu &\approx \frac{4\pi f_{\text{a}}^2\Delta}{3}\left[\dddot{X}^\mu - \left(\frac{\dot{X}\cdot\dddot{X}}{1-\dot{X}^2}\right)\dot{X}_\mu + \left(\frac{X'\cdot\dddot{X}}{1-\dot{X}^2}\right)X'_\mu\right].
\end{aligned} \tag{34}$$

2.6 Generalization to the temporal transverse gauge

For flat-space string dynamics, the conformal string gauge is usually employed. However, when considering problems in which the string energy decays, it more convenient to use the temporal transverse gauge in which $X^0 = t = \tau$ and $\dot{\mathbf{X}}.\mathbf{X}' = 0$ with $X^\mu = (t, \mathbf{X})$. In this gauge, the equations of motion for the string (8) are

$$\mu_0\left(\ddot{\mathbf{X}} - \frac{1}{\epsilon}\left(\frac{\mathbf{X}'}{\epsilon}\right)'\right) = \mathbf{f} \,, \quad \mu_0\dot{\epsilon} = f^0 \,, \tag{35}$$

where $\epsilon^2 = \dot{\mathbf{X}}^2/(1 - \mathbf{X}'^2)$ (not to be confused with the relative amplitude $\mathcal{E}$) and $\mathcal{F}^\mu = (f^0, \epsilon\mathbf{f} + f^0\dot{\mathbf{X}})$. Radiative damping will naturally be incorporated in the decay of the coordinate energy density ϵ, rather than in the non-intiutive time redefinitions of the

conformal gauge. The string energy and momentum per unit length in the temporal transverse gauge are then given by

$$E = \frac{\mu_0}{L} \int_0^L d\sigma \epsilon \,, \quad \mathbf{p} = \frac{\mu_0}{L} \int_0^L d\sigma \epsilon \dot{\mathbf{X}} \,. \tag{36}$$

The renormalisation procedure in this gauge is similar to that for the conformal gauge, but with the added complication that $\dot{X}^2 + X'^2 \neq 0$. After a detailed set of manipulations one can deduce that

$$
\begin{aligned}
H_{\mu\alpha\beta}^{\text{self}} =& \frac{f_{\mathrm{a}}}{2\dot{X}^4} \left[\frac{1}{\epsilon} \ddot{X}_{[\mu} V_{\alpha\beta]} - \frac{1}{\epsilon^3} X''_{[\mu} V_{\alpha\beta]} \right] \log(\Delta/\delta) + \mathcal{O}(\Delta^2) \,, \\
H_{\mu\alpha\beta}^{\text{rad}} =& \frac{f_{\mathrm{a}}}{2\dot{X}^4} \left[-\frac{4}{3} \dddot{X}_{[\mu} V_{\alpha\beta]} - \frac{1}{2} \dot{X}_{[\mu} \dot{V}_{\alpha\beta]} + \frac{3\dot{\epsilon}}{2\epsilon} X''_{[\mu} V_{\alpha\beta]} \right. \\
& \left. + \left(\frac{2\dot{X}.\ddot{X}}{\dot{X}^2} - \frac{\dot{\epsilon}}{2\epsilon} \right) \ddot{X}_{[\mu} V_{\alpha\beta]} \right] \Delta + \mathcal{O}(\Delta^2) \,,
\end{aligned}
\tag{37}
$$

where $A_{[\mu\alpha\beta]} = A_{\mu\alpha\beta} + A_{\beta\mu\alpha} + A_{\alpha\beta\mu}$. Note that, once again, the self-field has no order Δ term. Ignoring terms of order Δ^2, we can then deduce expressions for the self-force and the radiation backreaction force,

$$
\begin{aligned}
\mathbf{f}^{\text{self}} =& -2\pi f_{\mathrm{a}}^2 \log(\Delta/\delta) \left(\ddot{\mathbf{X}} - \frac{1}{\epsilon} \left(\frac{\mathbf{X}'}{\epsilon} \right)' \right) \,, \\
\mathbf{f}^{\text{rad}} =& \pi f_{\mathrm{a}}^2 \Delta \left\{ \frac{4}{3} \epsilon \dddot{\mathbf{X}} + \left[2\epsilon \left(\frac{\dot{\mathbf{X}} \cdot \ddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right) + 3\dot{\epsilon} \right] \ddot{\mathbf{X}} - \frac{2}{\epsilon} \left(\frac{\mathbf{X}' \cdot \ddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right) \dot{\mathbf{X}}' - \frac{3\dot{\epsilon}}{\epsilon^2} \mathbf{X}'' \right. \\
& \left. + \left[-\frac{4}{3\epsilon} \left(\frac{\mathbf{X}' \cdot \dddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right) - \frac{4}{\epsilon} \frac{(\dot{\mathbf{X}} \cdot \ddot{\mathbf{X}})(\mathbf{X}' \cdot \ddot{\mathbf{X}})}{(1 - \dot{\mathbf{X}}^2)^2} - \frac{\dot{\epsilon}}{\epsilon^2} \left(\frac{\mathbf{X}' \cdot \ddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right) + \frac{3\dot{\epsilon}}{\epsilon^4} \left(\frac{\dot{\mathbf{X}} \cdot \mathbf{X}''}{1 - \dot{\mathbf{X}}^2} \right) \right] \mathbf{X}' \right\} \,, \\
f^{0,\text{self}} =& -2\pi f_{\mathrm{a}}^2 \log(\Delta/\delta) \dot{\epsilon} \,, \\
f^{0,\text{rad}} =& \pi f_{\mathrm{a}}^2 \Delta \left\{ \frac{4}{3} \epsilon^2 \left(\frac{\dot{\mathbf{X}} \cdot \dddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right) + 2 \left(\frac{\mathbf{X}' \cdot \ddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right)^2 + 2\epsilon^2 \left(\frac{\dot{\mathbf{X}} \cdot \ddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right)^2 \right. \\
& \left. + 3\epsilon\dot{\epsilon} \left(\frac{\dot{\mathbf{X}} \cdot \ddot{\mathbf{X}}}{1 - \dot{\mathbf{X}}^2} \right) - \frac{3\dot{\epsilon}}{\epsilon} \left(\frac{\dot{\mathbf{X}} \cdot \mathbf{X}''}{1 - \dot{\mathbf{X}}^2} \right) \right\} \,.
\end{aligned}
\tag{38}
$$

As in the conformal gauge, the expressions for $\mathbf{f}^{\text{self}}$ and $f^{0,\text{self}}$, facilitate the well-known renormalisation of the equations of motion (35) and coordinate energy density,

$$\mu(\Delta) \left(\ddot{\mathbf{X}} - \frac{1}{\epsilon} \left(\frac{\mathbf{X}'}{\epsilon} \right)' \right) = \mathbf{f}^{\text{rad}} \,, \quad \mu(\Delta) \dot{\epsilon} = f^{0,\text{rad}} \,, \tag{39}$$

where the expressions for $\mathbf{f}^{\mathrm{rad}}$ and $f^{0,\mathrm{rad}}$ represent the finite radiation backreaction force. The renormalised versions of (36) are

$$E = \frac{\mu(\Delta)}{L} \int_0^L d\sigma\epsilon \,, \quad \mathbf{p} = \frac{\mu(\Delta)}{L} \int_0^L d\sigma\epsilon\dot{\mathbf{X}} \,. \tag{40}$$

Differentiating (40), gives the power and force due to radiation backreaction

$$\begin{aligned} \dot{E} &= \frac{\mu(\Delta)}{L} \int_0^L d\sigma\dot{\epsilon} = \frac{1}{L} \int_0^L d\sigma\, f^{0,\mathrm{rad}} \,, \\ \dot{\mathbf{p}} &= \frac{\mu(\Delta)}{L} \int_0^L d\sigma \left[\epsilon\ddot{\mathbf{X}} + \dot{\epsilon}\dot{\mathbf{X}} \right] = \frac{1}{L} \int_0^L d\sigma \left[\epsilon\mathbf{f}^{\mathrm{rad}} + f^{0,\mathrm{rad}}\dot{\mathbf{X}} \right] \,. \end{aligned} \tag{41}$$

Again, some of the terms in (37) and (38) can be shown to be sub-dominant, as for the conformal gauge. In particular it is possible to approximate $H_{\mu\alpha\beta}^{\mathrm{rad}}$, $\mathbf{f}^{\mathrm{rad}}$ and $f^{0,\mathrm{rad}}$ by

$$\begin{aligned} H_{\mu\alpha\beta}^{\mathrm{rad}} &\approx -\frac{2f_{\mathrm{a}}\Delta}{3\dot{X}^4} \dddot{X}_{[\mu} V_{\alpha\beta]} \,, \\ \mathbf{f}^{\mathrm{rad}} &\approx \frac{4\pi f_{\mathrm{a}}^2\Delta}{3} \left[\epsilon\dddot{\mathbf{X}} - \frac{1}{\epsilon}\left(\frac{\mathbf{X}'\cdot\ddot{\mathbf{X}}}{1-\dot{\mathbf{X}}^2} \right)\mathbf{X}' \right] \,, \\ f^{0,\mathrm{rad}} &\approx \frac{4\pi f_{\mathrm{a}}^2\Delta}{3} \left[\frac{\epsilon^2\dot{\mathbf{X}}\cdot\dddot{\mathbf{X}}}{1-\dot{\mathbf{X}}^2} \right] \,. \end{aligned} \tag{42}$$

Substituting the expressions for $f^{0,\mathrm{rad}}$ into the power expression (41) yields

$$\frac{dP}{dl} = -\dot{E} = -\frac{4\pi f_{\mathrm{a}}^2\Delta}{3L} \int_0^L d\sigma \frac{\epsilon^2\dot{\mathbf{X}}.\dddot{\mathbf{X}}}{1-\dot{\mathbf{X}}^2} \,. \tag{43}$$

2.7 Eliminating 'runaway' solutions

This simplified form of the equations of motion using (42) still has serious shortcomings because of the presence of the $\dddot{\mathbf{X}}$ term. The equations have unphysical, exponentially growing or 'runaway' solutions which will, for example, plague any potential numerical applications. Furthermore, one would be required to store information at three different timesteps, fundamentally changing the nature of a numerical algorithm. It appears, however, that both these problems can be circumvented by resubstituting the equations of motion, that is, we make the approximations $\ddot{\mathbf{X}} \approx \epsilon^{-1}(\mathbf{X}'/\epsilon)'$ and $\dddot{\mathbf{X}} \approx \epsilon^{-1}(\dot{\mathbf{X}}'/\epsilon)'$ in (42) (note we have used the unperturbated equations with $\dot{\epsilon} \approx 0$). The equations of

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motion then acquire an analogue of a viscosity term for which there are only damped solutions. After performing this resubstitution one finds that the approximate force (42) becomes

$$
\mathbf{f}^{\text{rad}} \approx \frac{4\pi f_{\text{a}}^2 \Delta}{3} \left[\left(\frac{\dot{\mathbf{X}}'}{\epsilon} \right)' - \frac{1}{\epsilon^2} \left(\frac{\mathbf{X}' \cdot (\dot{\mathbf{X}}'/\epsilon)'}{1 - \dot{\mathbf{X}}^2} \right) \mathbf{X}' \right],
$$
$$
f^{0,\text{rad}} \approx \frac{4\pi f_{\text{a}}^2 \Delta \epsilon}{3} \left[\frac{\dot{\mathbf{X}} \cdot (\dot{\mathbf{X}}'/\epsilon)'}{1 - \dot{\mathbf{X}}^2} \right].
$$
(44)

The reason for the suppression of the exponentially growing solution becomes apparent if we consider simplified one-dimensional equations,

$$
\dddot{X} - X'' = \alpha \dddot{X} , \quad \longrightarrow \quad \ddot{X} - X'' \approx \alpha \dot{X}'' ,
$$
(45)

where we have performed the resubstitution assuming that α is small. We now take an approximately periodic solution, $X'' \approx -\Omega^2 X$, and we substitute the ansatz $X \sim e^{mt}$. The solutions for (45) are given respectively by the roots of the following polynomials in m,

$$
f(m) = \alpha m^3 - m^2 - \Omega^2 , \quad \longrightarrow \quad g(m) = -m^2 - \alpha \Omega^2 m - \Omega^2 .
$$
(46)

Figure 4: A schematic of the contributions to the radiation backreaction force for a perturbed long string configuration. (a) The force is calculated by integrating along σ, that is, summing the contributions of all string segments on the backward light cone. (b) The expected appearance of the actual radiation force contributions; this has a local maximum and a fairly rapid fall-off with cancellations. Here, L is the typical wavelength of perturbations on the string (or the string curvature radius) and the area under the curve is the total magnitude of the force. (c) The radiation force is estimated in the local backreaction approximation using the local magnitude of the force and an effective width Δ for which the area under the two curves is equal. For this approximation to be valid, Δ must be less than the string curvature radius.

If we rewrite $f(m) = (m^2 + Am + \Omega^2 + B)(\alpha m - C)$, then we see that $A = \alpha\Omega^2 + \mathcal{O}(\alpha^2)$, $B = \mathcal{O}(\alpha^2)$ and $C = 1 + \mathcal{O}(\alpha^2)$. If we ignore terms $\mathcal{O}(\alpha^2)$, then the solutions of $g(m) = 0$ are approximately solutions of $f(m) = 0$. However, the real positive solution of $f(m) = 0$, corresponding to the exponentially growing solution of the equations of motion, is not a solution of $g(m) = 0$. Fig. 3 shows the relative positions of the curves $y = f(m)$ and $y = g(m)$ for typical values of the parameters Ω and α.

2.8 Understanding the 'local backreaction approximation'

The local backreaction approximation effectively reduces the problem of calculating the backreaction force from the string to the equivalent problem for the electron. The assumptions underlying (32) and (39) are that the dominant contributions to the integrals (27), (28) and (30) come from string segments close to the point under consideration. In fig. 4 we have schematically illustrated the construction of the radiation backreac-

tion force in the local backreaction approximation. To calculate this force at the time t at a particular point on the string (say $\sigma = 0$), we must sum over the retarded time contributions from all other string segments. For definiteness, let us suppose we are considering a long straight string with perturbations of typical wavelength L (comparable to the string curvature radius R). As we integrate along the backward light cone of fig. 4a, we can expect force contributions to take a form appearing something like fig. 4b. The precise rate of the fall-off of this force density away from $\sigma = 0$ is unknown, but finiteness certainly implies that it is faster than $1/\sigma$. Moreover, regions of the string beyond the curvature radius R will give negative, as well as positive, contributions and the resulting net cancellations should ensure rapid convergence of the integrated force at large distances.

The total area under the curve in fig. 4b represents the exact magnitude of the radiation backreaction force. The 'local backreaction approximation' to this force is illustrated in fig. 4c. We calculate the actual magnitude of the force at the point in question and we assume that the contribution from neighbouring segments falls away rapidly beyond an effective width Δ. We then normalize Δ to ensure that the area under the curve (c) is equal to that under curve (b). In §3 we shall discuss the procedure for achieving this by comparing with analytic and numerical radiation calculations.

Of course, we do not expect our local force to evolve every string trajectory completely accurately. We are, after all, assuming a uniform Δ, though it is possible to improve this first approximation. Furthermore, the force given by (42), which we shall use in the numerical simulations of §3, breaks worldsheet covariance and so we should anticipate difficulties describing some special 'null' string trajectories. This is because we have taken time as a preferred direction in the derivation of the Lienard-Wiechart potentials. Effectively, we calculate the backreaction force by summing up all contributions from points on the string inside the region D given by

$$D = \left\{ (\sigma, \tau) \text{ s.t. } -\Delta/2 < \sigma < \Delta/2 \,, -\Delta/2 < \tau < \Delta/2 \right\}, \qquad (47)$$

which corresponds to the shaded diamond in fig. 3a. However, there exists a family

of elongated rectangular regions of equal area related to D by Lorentz boosts. If the string trajectory has a typical wavelength (or periodicity) then it seems likely that the contributions from the two regions will be similar and the inaccuracies in the force should cancel out. We anticipate, therefore, that errors in the local backreaction approximation should be small for generic long string trajectories and for closed loops where this 'pseudo-periodicity' is likely to be evident.

In essence the validity of the local backrection approximation hinges on whether the following integral is small:

$$\partial_\mu B_{\alpha\beta}(x) = -\frac{f_a}{2} \int_{|\sigma-\bar\sigma|>\Delta/2} d\bar\sigma \frac{1}{\Delta \cdot \dot X} \frac{\partial}{\partial\bar\tau} \left(\frac{\Delta_\mu V_{\alpha\beta}}{|\Delta \cdot \dot X|} \right) \Bigg|_{\bar\tau=\tau_R}. \tag{48}$$

To summarize, we believe that (48) can be neglected for strings in a realistic network because the natural long distance force fall-off will be augmented by strong cancellations from a random superposition of distant modes. In any case, at the very least, this approximation should work in an some 'averaged' sense.

2.9 Analytic models for radiative decay

We can begin to develop confidence in the veracity of the local backreaction approximation by demonstrating that it predicts the correct scale-dependence of the overall radiation power from closed loops and long string trajectories. By analogy with the simple model for backreaction of ref. [5], we shall deduce expressions for the evolution of the invariant string length L for loops and the relative amplitude $\mathcal{E}$ for perturbed long strings, using some of the solutions we presented in §2.2.

(i) Closed loop solutions

For closed loops of invariant length L, we can estimate that $\dot{\mathbf{X}} \sim \mathcal{O}(1)$ and $\dddot{\mathbf{X}} \sim \mathcal{O}(L^{-2})$ (the actual average over one period is $\langle|\dot{\mathbf{X}}|\rangle = 1/\sqrt{2}$). If we take $\Delta \sim L$ in our local approximation, then the power per unit length (43) is proportional to L^{-1}, which immediately recovers the well known result that the power loss from a loop is independent of its size L [14]. In general, one can write the radiation power as

$$P = \Gamma_a f_a^2 = \kappa\mu, \tag{49}$$

where Γ_a is some factor dependent on the particular loop trajectory, but not its size and κ is the radiation backreaction scale, assumed to be independent of time†. The radiation damping can be modelled by considering the following equations,

$$
E = \mu \int_0^L d\sigma\epsilon = \mu L \,,
$$
$$
P = -\frac{dE}{dt} = \Gamma_a f_a^2 = \kappa\mu \,.
\tag{50}
$$

Integrating these equations one obtains,

$$
L = L_0 - \kappa(t - t_0) \,.
\tag{51}
$$

(ii) Long string solutions

The generic result for a long string solution, parametrized by its wavelength L and relative amplitude $\mathcal{E}$, is $\dot{\mathbf{X}} \sim \mathcal{O}(\mathcal{E})$ and $\dddot{\mathbf{X}} \sim \mathcal{O}(\mathcal{E}L^{-2})$. If $\Delta \sim L$, then the power per unit length in (43) is

$$
\frac{dP}{dl} = \frac{\beta\mathcal{E}^2}{L} \,,
\tag{52}
$$

where $\beta \sim f_a^2$ quantifies the overall strength of the radiation. By analogy to the closed loops the radiation damping can be modelled by,

$$
E = \frac{\mu}{L} \int_0^L d\sigma\epsilon = \mu + \alpha\mu\mathcal{E}^2 \,,
$$
$$
\frac{dP}{dl} = -\frac{dE}{dt} = \frac{\beta\mathcal{E}^2}{L} \,,
\tag{53}
$$

where α is the (order unity) solution-dependent coefficent of $\mathcal{E}^2$ in the power series expansion of

$$
\left[\frac{1}{L} \int_0^L \frac{dX^3}{d\sigma} d\sigma \right]^{-1} .
\tag{54}
$$

The power loss (52) will lead to an exponential decay of the amplitude and oscillation energy per unit length E,

$$
\mathcal{E} = \mathcal{E}_0 \exp\left(-\frac{\beta t}{2\alpha\mu L} \right), \quad E = \mu + \alpha\mu\mathcal{E}_0^2 \exp\left(-\frac{\beta t}{\alpha\mu L} \right).
\tag{55}
$$

---

† Problems involving global strings become intractable if the logarithmic time dependence of the string tension is included. In cosmological problems one finds that the logarithm changes by only about one order of magnitude over the enormous timescale between string formation and the present day. We will make this assumption in all the following calculations.

Exponential decay has already been shown to be generic [5] for a realistic situation where the left- and right-moving amplitudes are not precisely the same as in (15). However, for some of the periodic solutions of §2.2 such as the 'standing wave' solutions (14) and (17), the amplitude fall-off was shown analytically to be a power law, a fact also confirmed in numerical field theory simulations [5]. How can we reconcile this apparent contradiction? The answer lies in noting that the local backreaction approximation only applies to string configurations in which long-range field correlations are suppressed beyond the string curvature radius. Thus when we compare the effects of the radiation backreaction force with numerical field theory simulations, we must ensure that global correlations are suppressed, eliminating artificial situations with large-scale periodic coherence.

3 Numerical Comparisons

3.1 Numerical methods

In order to solve the modified equations of motion (39) and (44) numerically, one must recast the simplified resubstituted equations of motion into a first-order form accessible to numerical solution. Defining $\vec{\alpha} = \mathbf{X}' - \epsilon\dot{\mathbf{X}}$ and $\vec{\beta} = \mathbf{X}' + \epsilon\dot{\mathbf{X}}$, the equations of motion can be rewritten as

$$\mu(\Delta)\left[\dot{\vec{\alpha}} + \left(\frac{\vec{\alpha}}{\epsilon}\right)'\right] = -\frac{1}{2}\left(\epsilon\mathbf{f}^{\mathrm{rad}} + f^{0,\mathrm{rad}}\dot{\mathbf{X}}\right),$$

$$\mu(\Delta)\left[\dot{\vec{\beta}} - \left(\frac{\vec{\beta}}{\epsilon}\right)'\right] = \frac{1}{2}\left(\epsilon\mathbf{f}^{\mathrm{rad}} + f^{0,\mathrm{rad}}\dot{\mathbf{X}}\right), \tag{56}$$

$$\mu(\Delta)\dot{\epsilon} = f^{0,\mathrm{rad}}, \qquad \dot{\mathbf{X}} = \frac{1}{2\epsilon}\left(\vec{\beta} - \vec{\alpha}\right).$$

Using the above, we were able to evolve string trajectories by modifying a total variation non-increasing (TVNI) algorithm [22,9] which has already been well-tested for string network evolution in an expanding universe. This method relies on the fact that the first order equations of motion (56) are in conservative form, if the backreaction force is zero. Artificial compression methods are used to prevent the numerical dissipation of kinks. Typically, the algorithm maintains the perpendicularity condition $\dot{\mathbf{X}} \cdot \mathbf{X}' = 0$ and

conserves energy to within a few percent over many timesteps. It should be noted that the addition of a small backreaction forcing term does not seem to affect the stability of the numerical scheme. However, if the backreaction force becomes larger than the tension force, then the equations of motion become qualitatively different behaving like a diffusion equation rather than a hyperbolic wave equation. The characteristic Courant condition for a diffusion equation is very much more restrictive and so stability problems will emerge in this regime. One can address this numerical problem by artificially preventing the force from becoming too large, that is, the usual procedure of 'force softening'. There is a further technical numerical problem because the coordinate energy ϵ decays more rapidly at certain points (for example, at the cusps of the Kibble–Turok loops). Eventually, this imposes an unacceptably small timestep on the simulation because we always require $\Delta t < \epsilon \Delta \sigma$ everywhere. However, this problem can be solved by a number of approaches, including multiple time-stepping in small ϵ regions, by reparametrising the string to redistribute ϵ more evenly, or by eliminating such regions through 'point-joining' techniques.

In all the numerical simulations using the radiation backreaction force in this paper, we have employed the constant damping coefficent given by

$$\frac{4\pi f_{\mathrm{a}}^2 \Delta}{3\mu(\Delta)} \approx 0.001L \,. \tag{57}$$

This choice of damping coefficient corresponds to the cosmologically interesting paremeter range $\mu(\Delta) \sim 100 f_{\mathrm{a}}^2$ with our numerically determined normalization $\Delta \sim 0.1L$ which we shall discuss in the next section. When we compare the results of these Nambu string simulations with those using the underlying field theory for which $\mu(\Delta) \sim 5 f_{\mathrm{a}}^2$, we have had to perform a single global rescaling of the time axis in order to take into account the different radiation strengths.

As well as this one-dimensional effective model, we have developed sophisticated numerical algorithms to dynamically simulate string configurations in the Goldstone model (2) [5,23,24]. We discretize space on a three-dimensional grid with dimensions N_1, N_2, N_3 in the x, y, z directions respectively, solving the rescaled ($f_a \to 1, \ \lambda \to 2$)

Euler-Lagrange equation,

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial z^2} + \Phi(\bar{\Phi}\Phi - 1) = 0 \,. \tag{58}$$

We employ a second-order leapfrog algorithm for the time derivative and fourth-order finite difference approximations for the spatial derivatives. In problems where radiation is incident on the boundaries, its sensible to use absorbing boundary conditions [5,24]. A second order wave equation which annihilates the reflected wave at the $x = 0$ boundary is

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial x} - \frac{\partial^2 \Phi}{\partial^2 t} + \frac{1}{2}\left(\frac{\partial^2 \Phi}{\partial^2 y} + \frac{\partial^2 \Phi}{\partial^2 z}\right) = 0 \,. \tag{59}$$

The efficacy of these methods is discussed in some detail in ref. [5].

We use a cylindrically symmetric string ansatz to create initial conditions for both long string and loop solutions as in ref. [5]. However, a naive application of this ansatz artificially creates long-range correlations which do not conform with the assumptions underlying the 'local backreaction approximation'. As we have emphasised, general configurations that occur in realistic string networks will not have field correlations beyond the average curvature radius of the string because of reconnection processes and causality constraints. Consequently, we have modified our ansatz for long string configurations by numerically suppressing the initial perturbations with the gaussian,

$$e^{-(|\mathbf{x}-\mathbf{x}_\mathrm{s}|/R)^2} \,, \tag{60}$$

where R is the string curvature radius, $\mathbf{x}$ is the position in question and $\mathbf{x}_\mathrm{s}$ is the nearest long string segment. At large distances $r \gg R$, therefore, the string fields will approach those for a straight string, as we would expect in a general physical context for random small-scale structure.

To make this distinction plain, fig. 5 illustrates the effect of large distance correlation suppression on the decay of a periodic sinusoidal solution. The suppressed case (fig. 5b) can be seen initially to decay more rapidly than the configuration in which perturbations in the fields are correlated out to the simulation boundary (fig. 5a); the

Figure 5: A comparison of the decay amplitude for a sinusoidal solution in full field theory (a) without suppression and (b) with suppression of the field at the curvature radius. Notice that the initial decay rate of the suppressed configuration is much faster (exponential) than that for the unsuppressed configuration (power law) due to the long range correlations of the latter.

former is exponential decay, while the latter is power law. However, given the periodic boundary conditions, this difference does not persist indefinitely because the suppressed configuration will causally relax to an unsuppressed one, as the long range fields become correlated on larger and larger scales. Given this limitation imposed by the numerical grid size, we can only expect to normalize the local backreaction approximation using relatively short simulations (or by using non-periodic configurations).

3.2 Long string configurations

We have extensively tested the 'local backreaction approximation', using the modified

Figure 6: Log-linear plots for the decay of amplitude for (a) a sinusoidal solution, (b) a pure left moving helicoidal perturbation, and (c) a helicoidal perturbation with unequal left- and right-moving amplitudes. The straight line typifies the exponential decay.

Nambu equations of motion (56) and the approximate force derived from (42) by directly comparing with field theory simulations of radiating strings in the Goldstone model.

Using our modified Nambu string simulations, we find that exponential decay is generic for all the long string configurations discussed in §2. This decay is illustrated in fig. 6 for the sinusoidal solution (17), the pure right-moving helicoidal solution (16) and the helicoidal solution with unequal left- and right-moving amplitudes (15). Fig. 7 illustrates the excellent quantitative agreement with the full field theory simulations by

Figure 7: Decay of $\mathcal{E}$ using the radiative backreaction force (dotted line) and numerical field theory simulations (solid line) for (a) a sinusoidal perturbation, (b) a helicoidal perturbation with unequal left- and right-moving amplitudes, and (c) a pure left moving helicoidal perturbation. Note the excellent quantitative agreement for all three cases.

direct comparison with the same three long string solutions. Note that these curves have not been matched separately; the same backreaction damping coefficient applies for each and there has been only a single global rescaling. The agreement persists for the longest time for the (generic) unequal left- and right-moving configuration (15) because exponential decay is predicted in this case even after field correlations have relaxed at large distances. By comparing with the simple backreaction model for exponential decay (55), one can use the numerical field theory results to normalize Δ, that is, we estimate

$$\Delta \approx (0.1 \pm 0.02)L \qquad (61)$$

for the long string solutions investigated. This is a result for which there are considerable uncertainties at this stage, mainly because of the imprecision inherent in our small-scale field theory simulations. We had anticipated that Δ should be normalized to a distance near the string radius of curvature R, which for a sinusoidal perturbation is $R \sim L/4$. The fact that Δ is smaller than R validates the linearized expansion on which the 'local backreaction approximation' is based. The normalisation of the value of Δ above can become ambiguous in certain physical contexts, such as a solution with a number of different Fourier modes. In this case, we must make a further approximation by normalising to the lengthscale which is radiatively dominant.

We have also applied these numerical approaches to study the kink solution (19) of §2. Fig. 8 compares the evolution of a sharp kink in both the local backreaction approximation and in a field theory simulation. Backreaction leads to a substantial rounding of the kink, in agreement with the intuitive picture described in refs. [25,5]. The results are almost indistinguishable except for the computational advantages of the former which, in this case, saved a factor of 10^2 in cpu time and 10^4 in allocated memory.

We have also performed spectral analysis of the modes on the string using techniques similar to those used in three dimensions in ref. [5]. The kink itself can be written as an infinite series of odd Fourier modes, while the anticipated endpoint, a sinusoidal solution, is just a single Fourier mode. Fig. 9 illustrates the mode decomposition of the kink solution initially and then at late times. Radiation backreaction causes decay in all modes, but the higher harmonics are clearly damped much more strongly, leading to the kink 'rounding'. These spectra can be compared to the kink radiation fields shown in ref. [5] which demonstrate the same trend.

3.3 Closed loop solutions

We have also applied the local backreaction approximation to the study of loop solutions, such as the Kibble-Turok loops described in §2.2. In this case, $\Delta \sim L$ is not independent of time because the loop shrinks as it decays*. This problem can be circumvented in the

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* For the long string solutions, periodic boundary conditions forced the solution to have a fixed time-

Figure 8: Decay of a kink perturbation $(\mathcal{E}_0{=}0.9)$ using (a) the radiation backreaction force and (b) numerical field theory. Notice the visible rounding of the kink in both cases.

case of loops by choosing the cut-off scale Δ equal to a constant multiple of the total invariant string length,

$$L = \int_0^{2\pi} d\sigma\, \epsilon\,, \qquad (62)$$

which is easily calculable within the evolution algorithm described earlier.

The damped evolution of the special kinky loop solution (12) is shown after several oscillations in fig. 10. As the loop shrinks in size, there is discernible 'rounding' due to the radiative damping, though it is less pronounced than in the long string kink decay. Unfortunately, evolution for this and other loops could not be continued indefinitely because a numerical solution to the Courant violation problem in small ϵ regions has yet to be implemented (refer to §3.1). However, the observed 'rounding' is at least qualitatively in agreement with a previous attempt to study gravitational backreaction in ref. [26]. In

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independent wavelength $\lambda \sim L$.

Figure 9: The time evolution of the Fourier modes of an initial kink configuration (a). Note the damping of higher modes after 16 oscillations in (b).

Figure 10: The time evolution of a kinky loop solution shown initially and after 5 and 10 oscillations. Note the decrease in loop size and discernible kink 'rounding'.

Figure 11: The effect of radiative damping on loop energy for $\alpha = 0$ Kibble–Turok loops with $\phi = \pi/12$ (solid line), $\pi/3$, (dotted line) and $5\pi/12$ (dashed line). Note the expected linear decay of the loop length.

this non-local approach, all the retarded time radiation contributions were accumulated for an unperturbed loop trajectory and then these 'corrections' were applied at the end of each oscillation period. Unlike the local backreaction approximation, there is little prospect of such 'exact' approaches being implemented in network simulations because the $\mathcal{O}(N^2)$ algorithms require a supercomputer to evolve a single loop. Nevertheless, we anticipate future quantitative comparisons with such methods to determine the accuracy of our approach.

The evolution of the energy of some Kibble–Turok loops is illustrated in fig. 11. One can readily observe the linear decay of these solutions, as expected from our simple backreaction model (51). Notice, however, the oscillatory nature of the decay due to stronger radiation when the loop trajectories becomes more convoluted and when cusps

appear. It is interesting to note that a preliminary analysis indicates that, while cusp velocities are curtailed by backreaction, their periodic reappearance in these particular solutions is not actually prevented.

The overall decay rate in fig. 11 is parameter dependent; the slope yields the backreaction scale κ (or Γ_a) in (49) which is appropriate for the particular loop trajectory. Fig. 12 illustrates the ϕ-dependence of κ for the $\alpha = 0$ loop solutions (11). This is qualitatively similar to analytic estimates of the radiation from these loops, though at this stage we can only compare to results for gravitational radiation [14]. Note, however, that the divergences at small and large ϕ become weakened relative to the previous analysis; this may reflect a shortcoming of our approximation or the genuine influence of backreaction. However, a considerably more detailed quantitative analysis is necessary to test the accuracy of this approach, especially if we are to normalize it properly for string network simulations.

A study of the overall Kibble–Turok loop parameter space (which previous analyses have regarded as fairly typical) yielded an approximate value $\kappa \approx 0.1$, given the assumed damping coefficient (57) which was set by normalizing Δ with the long string results. However, $\kappa \approx 0.1$ is the typical backreaction scale expected for GUT-scale global string loops with $\mu(\Delta) \approx 100 f_a^2$ (refer to ref. [14,10]), thus independently validating our previous normalization

$$\Delta \approx 0.1L \, . \tag{63}$$

Given that the case for the local backreaction approximation is not as clear-cut for closed loop solutions, these results must be regarded as encouraging. At the very least, this approach can be used to phenomenologically incorporate expected loop decay rates, but results to date suggest it will do substantially better.

4 Radiative backreaction in an expanding universe

Our current understanding of the evolution of a cosmic string network is based on a marriage between analytic models and sophisticated network simulations [8,9]. However, the

Figure 12: Parameter dependence of the backreaction scale κ for $\alpha = 0$ Kibble–Turok loops using the local backreaction approximation.

network simulations only evolve the free equations of motion for a string in an expanding universe. In order to incorporate the radiative effects discussed in the preceding section one must modify the equations of motion to include a radiation damping term,

$$
\begin{aligned}
\mu_0 \left[\ddot{\mathbf{X}} + \frac{2\dot{a}}{a}(1 - \dot{\mathbf{X}}^2)\dot{\mathbf{X}} - \frac{1}{\epsilon}\left(\frac{\mathbf{X}'}{\epsilon}\right)' \right] &= \mathbf{f} \,, \\
\mu_0 \left[\dot{\epsilon} + \frac{2\dot{a}}{a}\epsilon\dot{\mathbf{X}}^2 \right] &= f^0 \,,
\end{aligned}
\tag{64}
$$

where a is the scale factor.

However, to calculate this radiation damping term one must use Green functions in an expanding background. In the radiation-dominated era, the retarded Green function is

$$
D_{\mathrm{ret}}(x, x') = \frac{a(\eta)}{2\pi a(\eta')}\delta\big((x - x')^2\big)\theta(\eta - \eta') \,.
\tag{65}
$$

where $x^\mu = (\eta, \mathbf{x})$ and $x'^\mu = (\eta', \mathbf{x}')$. In the matter-dominated era the retarded Green function also includes the effects of back-scatter off the background spacetime curvature,

$$D_{\text{ret}}(x, x') = \frac{a(\eta)}{2\pi a(\eta')} \left[\delta\big((x - x')^2\big)\theta(\eta - \eta') + \frac{1}{2\eta\eta'}\theta(\eta - \eta' - |\mathbf{x} - \mathbf{x}'|) \right].$$ (66)

Applying, the 'local backreaction approximation' in either of these scenarios, one finds that the forcing terms are given by,

$$\mathbf{f}^{\text{rad}} = \mathbf{f}^{\text{rad}}_{\text{flat}} + \frac{\dot{a}}{a}\mathbf{g} + \mathcal{O}(1/t^3)\,,$$

$$f^{0,\text{rad}} = f^{0,\text{rad}}_{\text{flat}} + \frac{\dot{a}}{a}g^0 + \mathcal{O}(1/t^3)\,,$$

where the flat suffix denotes the flat space backreaction force given by (42) and $(g^0, \mathbf{g})$ is a correction to the force due to the expanding background. Notice that to eliminate 'runaway' solutions due to $\dddot{\mathbf{X}}$, we must now resubstitute the damped expanding universe equations of motion (64). Recall that perturbations with lengthscales $r \gg H^{-1}$ are essentially 'frozen' by Hubble damping. It is for the same reason that large-scale perturbations will not radiate, despite the high degree of string curvature on these lengthscales.

The forced equations of motion (64) can be averaged to derive equations for the evolution of the density of long strings (ρ_∞) and loops (ρ_{L}), under the influence of the expansion, Hubble damping and the radiation backreaction force. If one now inserts a term to take into account of loop production, the equations become

$$\dot{\rho}_\infty = -\frac{2\dot{a}}{a}(1 + \langle v^2 \rangle)\rho_\infty - \frac{c\rho_\infty}{L} - \frac{d\rho_\infty}{L}\,,$$

$$\dot{\rho}_{\text{L}} = -\frac{3\dot{a}}{a}\rho_{\text{L}} + \frac{c\rho_\infty}{L}\,,$$ (67)

where

$$d = d_0 + \frac{\dot{a}}{a}d_1 + \mathcal{O}(1/t^2)\,,$$

$\langle v^2 \rangle$ is the average string velocity, $d_0, d_1, ..$ are constants and c is a measure of the efficiency of loop production. Substituting $\rho_\infty = \mu\zeta/t^2$ and $L = \zeta^{-1/2}t$ into (67), one obtains

$$\frac{\dot{\zeta}}{\zeta} = \frac{1}{t}\left[2 - 2\beta(1 + \langle v^2 \rangle) - (c + d)\zeta^{1/2}\right]\,,$$ (68)

where β is determined by the scalefactor, $a \propto t^\beta$. Eqn (68) has an attractive fixed point, which corresponds to the scaling regime. If $d_i = 0$ for $i > 0$, then we have

$$c = \zeta^{-1/2}(1 - \langle v^2 \rangle) - d_0 \,.$$

In the case where d_1 is non-zero, one should observe transient effects in the scaling. However, for large times these effects will be negligible and the attractive fixed point is exactly that for $d_1 = 0$.

5 Conclusions

We have introduced, and we have endeavoured to justify, a new approach to the study of radiation backreaction on strings (and other extended objects and membranes). If our analysis and the supporting evidence is valid, then the 'local backreaction approximation' offers the hope of quantitative insight into the essentially intractable problem of radiative damping effects during string network evolution. It is appropriate, therefore, to summarize the main points in our discussion.

By exploiting the analogy with classical electrodynamics we have used Green function methods to separate the self- and radiation fields of a global string, using the former to renormalize the string tension. We then approximated the radiation force at a point on the string by an expansion in powers of a cut-off parameter Δ. The 'local backreaction approximation' to the radiation force, then, is the local force at the point in question multiplied by an effective width Δ beyond which the neighbouring string segment contributions become negligible. We normalize Δ in order to reproduce the actual radiation force in known situations and we have confirmed the self-consistency of the approximation by demonstrating that Δ is less than the string curvature radius. We then generalized this approach to the temporal transverse gauge, a convenient gauge for studying dissipative string processes. The final step was to remove unphysical 'runaway' solutions, which plague the analogous point-particle analysis, by resubstituting the string equations of motion.

We then tested a numerical implementation of the 'local backreaction approximation' by investigating a variety of long string and closed loop trajectories. We have directly compared the results with analytic radiation calculations and numerical field theory simulations, demonstrating a consistent normalization for Δ using these independent methods. This approach reproduces the correct scale-dependence of radiative effects and demonstrates satisfactory quantitative agreement for a wide variety of different solutions. There is clearly scope for a more detailed analysis of the accuracy of this approach and for addressing a number of outstanding issues. However, we have presented sufficient grounds for believing that the 'local backreaction approximation' is a significant step forward in the study of string radiative backreaction.

Acknowledgments

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